MODAL ANALYSIS OF LUMPED FLEXIBLE ACTIVE SYSTEMS (PART 1)

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Abstract

Simulation and prediction of eigen-frequencies and resonance problems for flexible structures is an important task in disciplines such as robotics and aerospace engineering. However, little effort seems to have been put into the problem dealing with modal analysis of mechatronic systems containing coupled flexible structures and control systems. When, for instance, designing a satellite tracking radar, it is crucial to be able to predict resonance in the radar system during normal working conditions. Resonance may lead to loss of satellite tracking accuracy and long term fatigue problems.

This paper addresses the theory of solving the eigenvalue problem for a simple one-degree-of-freedom system coupled with a single position feedback PD-controller. To test the theory, the nonlinear multi-disciplinary simulation software FEDEM has been used. This paper is planned as the first in a series of papers addressing modal analysis of active flexible multibody systems.

Keywords:
Modal analysis, eigenvalue problem, flexible multibody system, PD-controller.

1 Introduction

To optimize performance and reduce development costs of mechatronic products, it is very important to use virtual testing. During the later years, mechanical products have become increasingly complex and mechanical functionality has gradually been replaced by cheaper and smarter control (active) systems. Typical examples are active / adaptive car suspensions, cranes, robots, machining centers, airplanes and satellites.

Mechatronic systems are traditionally designed and tested in separate software systems since the underlying mathematics used to solve the subsystems are different. Control systems are often modeled as 1st order equation systems (state-space-formulation), while mechanical systems usually are modeled as 2nd order symmetrical equation systems. These subsystems are therefore traditionally solved decoupled by different equation solvers. This approach has several disadvantages:

- The subsystems become sub-optimized because the couplings between them are limited. Control systems are often modeled as lumped springs and dampers in the mechanical subsystem and mechanical components are simplified as lumped masses, inertias and amplifiers in the control subsystem. The couplings between them are established through iterations and interchanges of force and response variables. The performance of the combined mechatronic system can thus not be simulated and optimized with a satisfactory accuracy and efficiency.

- The mechanical system and the control system are mutually affected by each other. Changes in either of the systems will cause alterations in the other. This means that the two mathematical models must be updated separately, which is both time consuming and demands coordination and handling of different software versions between engineers from different departments.

- A decoupled model representation does not support calculations of eigenfrequencies and mode shapes (modal analysis), which give engineers vital information about the overall performance of a mechatronic system.

A literature survey performed by the authors indicated that little effort has been put into the problem dealing with modal analysis of mechatronic systems containing flexible structures, like robots, cranes, suspension and aerospace systems. However, the topic has been discussed in some papers and reports. In [1] it is shown that when combining passive mechanical springs and active piezoelectric springs, the total stiffness of the system is a sum of the stiffness from each of the springs, as can be expected based on basic theory of dynamics. In [2] it is shown that actuators can be controlled to act like virtual passive mechanical spring-damper elements using a velocity feedback PI-controller. In [3] it is shown that in contact motion force control, both the gain from a controller and the stiffness of the structure influences the natural frequency of the system. In [4] it is mentioned that a position feedback PD-controller is physically equivalent to a virtual spring and damper whose reference position is moving with a desired velocity.

This paper focuses on eigenfrequency analysis for a mechanical system with one degree of freedom, combined with a position feedback PD-controller. First, a basic description of the PD-controller is given. Next, different variants of the one-degree-of-freedom system combining the PD-controller and the mechanical system are described, and an equation for the eigenfrequency of these systems is given. Finally, results derived from the eigenfrequency equation for a total of six different scenarios are compared to experimental tests performed in the nonlinear multi-disciplinary simulation software FEDEM [5].

2 The PD-controller

Fig. 1 shows a simple block diagram used for describing a single-input single-output (SISO) feed-back control system:

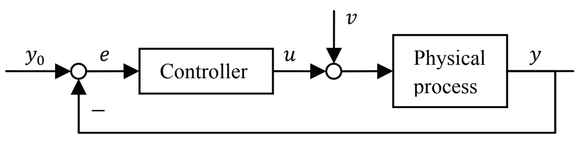


Fig. 1 Block diagram for a single-input single output (SISO) feedback control system.

The whole idea behind a control system is to manipulate a physical process to behave in a certain desired way. y_0 is the reference value for a parameter in the physical process and represents how this parameter should behave; y is the measured value for the same parameter and represents how this parameter actually is behaving. In a steady-state process, the aim is to keep the process as stable as possible, suppressing the disturbances v acting on the process as effectively as possible.

As shown in the block diagram, the difference e between the reference value y_0 and a measured value y is given by:

$$e = y_0 - y \tag{1}$$

e is somehow manipulated in the controller, and out comes a controller value u. This controller value is added with the disturbance v on the physical process. When combined, u and v make up the input value for the parameter in the physical process. The output value from the physical process is the measured value y, which is then compared to the reference value y_0, and the loop repeats itself.

The function of the controller is to manipulate the physical process so that it behaves in the most satisfactory way. One common approach to achieving this goal is to construct the controller with a combination of a proportional part (P), an integral part (I) and a derivative part (D). Based on the controller's incoming value $e(t)$, the outgoing controller value $u(t)$ for each of the three parts is, respectively:

$$u_P(t) = K_p e(t) \tag{2}$$

$$u_I(t) = K_i \int_0^t e(\tau) d\tau \tag{3}$$

$$u_D(t) = K_d \frac{de(t)}{dt} \tag{4}$$

where K_p, K_i and K_d are the proportional, integral and derivative gains, respectively.

If combined, the different parts give the following equation for the outgoing controller value $u(t)$:

$$u_{PID}(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \frac{de(t)}{dt} \tag{5}$$

If the integral gain K_i and derivative gain K_d are given by:

$$K_i = \frac{K_p}{T_i} \tag{6}$$

$$K_d = K_p T_d \tag{7}$$

then Eq. (5) can be written as:

$$u_{PID}(t) = K_p \left(e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt} \right) \tag{8}$$

The outgoing controller value $u(t)$ for a PD-controller, which only has a proportional and a derivative part, is:

$$u_{PD}(t) = K_p e(t) + K_d \frac{de(t)}{dt} \tag{9}$$

3 System with one degree of freedom

Fig. 2 illustrates three linear systems with one degree of freedom: a passive, an active and a coupled system. The passive system in Fig. 2 (a) is a mass-spring-damper system. The active system in Fig. 2 (b) contains only a mass and a controller. The coupled system in Fig. 2 (c) is a combination of the active and the passive system.

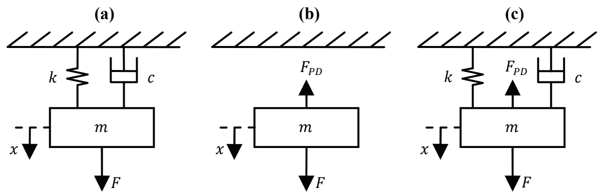


Fig. 2 Systems with one degree of freedom; (a) passive system, (b) active system, (c) coupled system.

All of the systems shown in Fig. 2 have the same degree of freedom: the position x of the mass m. Since these systems only have one degree of freedom, they will only have one eigenfrequency with one corresponding mode shape (oscillation).

3.1 Passive system

Fig. 3 illustrates a passive system with one degree of freedom. This is a system consisting of mechanical parts only.

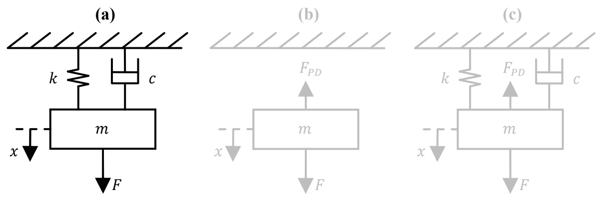


Fig. 3 Passive system with one degree of freedom.

The dynamic equation of motion for the passive system is:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \tag{10}$$

where m is the mass in kg, c is the damping coefficient in Ns/m, k is the spring stiffness in N/m and F is an external force in N acting on the system. x, $\dot{x}$ and $\ddot{x}$ are the position, velocity and acceleration of the mass, respectively.

$$\dot{x}(t) = \frac{dx(t)}{dt}, \ddot{x}(t) = \frac{d^2x(t)}{dt^2} \tag{11}$$

When calculating the eigenvalue of the system, all the external forces F are set to zero. The dynamic equation for the system thus becomes:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{12}$$

The eigenfrequency of an undamped system ($c = 0$) is then given by:

$$\omega_e = \sqrt{\frac{k}{m}} \tag{13}$$

If the system is damped, the eigenfrequency is given by [6]:

$$\omega_{e,d} = \omega_e\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m}} \cdot \sqrt{1-\zeta^2} \tag{14}$$

where ζ is the damping ratio ($\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}$).

3.2 Active system

Fig. 4 illustrates an active system with one degree of freedom. This system is almost identical to the passive system shown in Fig. 3. However, as a crucial difference, the spring and damper have been replaced by a control system, in this case a standard PD-controller.

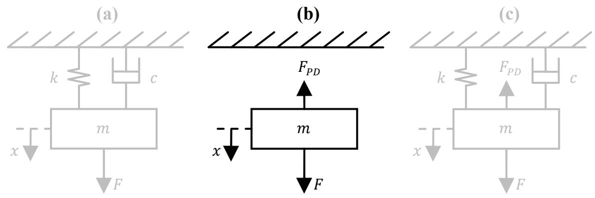


Fig. 4 Active system with one degree of freedom.

The dynamic equation of motion for the active system is:

$$m\ddot{x}(t) + 0\dot{x}(t) + 0x(t) = F(t) + F_{PD}(t) \tag{15}$$

where F_{PD} is the force from the PD-controller acting on the mass m.

When calculating the eigenvalues of the system, the dynamic equation has now become:

$$m\ddot{x} + 0\dot{x} + 0x = 0 \tag{16}$$

This implies that, from a classical mechanical point of view, the system does not have any eigenvalues since the elastic part of the dynamic equation is equal to zero. Thus the undamped eigenfrequency should be:

$$\omega_e = \sqrt{\frac{k}{m}} = \sqrt{\frac{0}{m}} = 0 \qquad (17)$$

However, when observing the active system, it is obvious that the system does have an eigenfrequency. A simple active system, in accordance to Fig. 4, was created and dynamically simulated in FEDEM, with no initial equilibrium iterations. The following figure illustrates the position of the mass m in the active system when under influence of a constant force F (equaling gravity). The mass was set to 1 kg and, for simplicity, the force was set to 10 N.

For the PD-controller, the proportional gain K_p was set to 100, the derivative gain K_d to 0 and the reference value y_0 representing the desired position of the mass to 0.

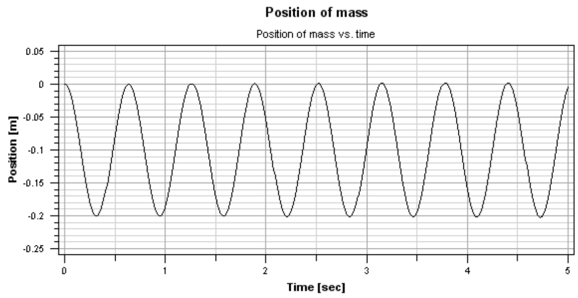


Fig. 5 Position of the mass in the active system with K_p = 100 and K_d = 0.

As Fig. 5 shows, the system clearly oscillates, even though the reference value y_0 is 0. This implies that the system does possess an eigenfrequency.

3.3 Coupled system

If the passive system in Fig. 3 is combined with the active system in Fig. 4 they form a coupled system, as shown in Fig. 6:

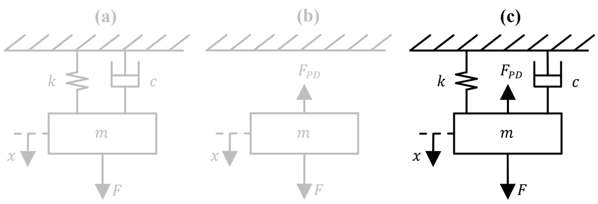


Fig. 6 Coupled system with one degree of freedom.

The dynamic equation of motion for the coupled system is:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) + F_{PD}(t) \qquad (18)$$

Fig. 7 shows a block diagram of the coupled system in Fig. 6:

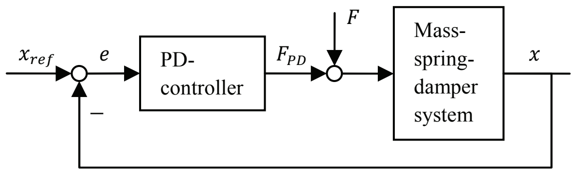


Fig. 7 Block diagram for a mass-spring-damper system and a PD-controller.

In the block diagram, x_{ref} is the reference position for the mass m. e is the difference between the reference position x_{ref} and the actual position x:

$$e = x_{ref} - x \qquad (19)$$

The PD-controller consists of a proportional part K_p and a derivative part $K_d \frac{d}{dt}$, as shown in Eq. (9). The outgoing value from the PD-controller is a force F_{PD}, which is added together with the force F acting on the mass object, and forms the basis for the incoming value to the mass-spring-damper system. The mass-spring-damper system is described by Eq. (18); its outgoing value is the position x of the mass m.

All of the forces acting on the system can be gathered into an equation containing the internal forces of the mechanical system on the left side and the external and controller forces on the right side:

$$F_m + F_c + F_k = F + F_P + F_D \qquad (20)$$

where F_m is the inertia force, F_c is the damping force, F_k is the elastic force, F is the external force, F_P is the force from the proportional part of the controller and F_D is the force from the derivative part of the controller. Eq. (20) can be rewritten to:

$$m\ddot{x} + c\dot{x} + kx = F + K_p e + K_d \dot{e} \qquad (21)$$

where $\dot{e}$ follows the same notation as given in Eq. (11). If Eq. (19) is inserted into Eq. (21), it can now be written as:

$$m\ddot{x} + c\dot{x} + kx = F + K_p(x_{ref} - x) + K_d(\dot{x}_{ref} - \dot{x}) \qquad (22)$$

which is equal to:

$$m\ddot{x} + c\dot{x} + kx + K_p x + K_d \dot{x} = F + K_p x_{ref} + K_d \dot{x}_{ref} \qquad (23)$$

or:

$$m\ddot{x} + (c + K_d)\dot{x} + (k + K_p)x = F + K_p x_{ref} + K_d \dot{x}_{ref} \qquad (24)$$

When calculating the eigenvalue of the system, the external force F and the reference position x_{ref} is set to zero, giving the following equation:

$$m\ddot{x} + (c + K_d)\dot{x} + (k + K_p)x = 0 \qquad (25)$$

The undamped and damped eigenfrequencies for the coupled system now becomes, respectively:

$$\omega_e = \sqrt{\frac{(k + K_p)}{m}} \qquad (26)$$

$$\omega_{e,d} = \sqrt{\frac{(k + K_p)}{m}} \cdot \sqrt{1 - \zeta^2} \qquad (27)$$

where the damping ratio ζ now has become:

$$\zeta = \frac{(c + K_d)}{2\left(\sqrt{(k + K_p)m}\right)} \qquad (28)$$

Eq. (26) and Eq. (28) correspond with formulas for effective natural frequency and effective damping ratio given in [6].

3.4 Experimental results

To verify the theoretical results derived above, a simple test was created. The objective of the test was to see how the mass-spring-damper system and the PD-controller actually acted on the eigenfrequency of the coupled system. The testing environment was created in the nonlinear multidisciplinary simulation software FEDEM, rather than using actual physical equipment. A total of 6 different testing scenarios were created:

<u>Undamped</u>

I) Only a spring connecting the mass to the ground.
II) Only a P-controller connecting the mass to the ground.
III) A spring and a P-controller connecting the mass to the ground.

<u>Damped</u>

IV) A spring-damper system and P-controller connecting the mass to the ground.
V) A spring and a PD-controller connecting the mass to the ground.
VI) A spring-damper system and a PD-controller connecting the mass to the ground.

An FE-model consisting of the coupled system shown in Fig. 6 was created in FEDEM. The external force F was set to 10 N and the reference position x_{ref} to 0. The mass m was set to 1 kg, the spring stiffness k to 100 N/m, the damping coefficient c to 7.2 Ns/m, the proportional gain K_p to 44 N/m and the derivative gain K_d to 4.8 Ns/m. k and K_p, and also c and K_d, have deliberately been given different values, such that differences are easier to distinguish. With respect to Eq. (26), Eq. (27) and Eq. (28), this should give the eigenfrequencies listed in Tab. 1 (calculations are shown in the appendix).

Fig. 8 and Fig. 9 show one graph from each of the six scenarios. The graphs picture the velocity of the mass m versus the time. The reason for using the velocity of the mass rather than its position is that it makes it easier to see the period of the oscillation. One period is then where the velocity is zero for the second time. Since FEDEM uses a numerical algorithm to solve the dynamic equation of motion, the results are only accurate to a certain number of decimals. To balance between accuracy of results and simulation running time, the time increment in the simulations was set to 0.0005 seconds.

4 Discussion

The results presented in Fig. 8 and Fig. 9 show that the eigenfrequencies from the simulations corresponds perfectly to the pre-calculated eigenfrequencies for the six different scenarios. As these results imply, the proportional gain K_p and derivative gain K_d from the PD-controller influences the stiffness and damping properties of the mechanical system, respectively. So, when performing an eigenvalue analysis for a mechanism coupled with a PD-controller, the proportional and derivative gain from the PD-controller should somehow be added to the stiffness and damping properties of the system. One way of doing this is to add a virtual spring with spring stiffness k_P corresponding to K_p and a virtual damper with damping coefficient c_D corresponding to K_d to the mechanical model. Another approach is to establish and solve the eigenvalues using a set of coupled equations representing the mechatronic system (mechanical and control system). This approach will be developed and reported in later papers, with a mission to solve eigenfrequencies and mode shapes for active flexible systems.

5 Conclusion

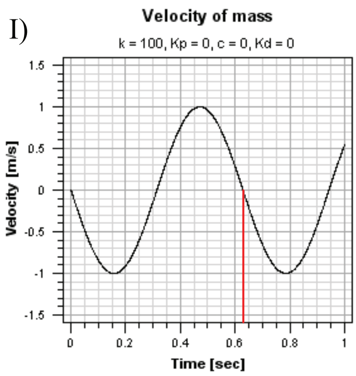
In this paper, a brief study of the eigenfrequencies of an active system containing a mass-spring-damper system and a position feedback PD-controller has been conducted. Theory for modal analysis of such a system has been derived and presented. The theory has been verified by experiments conducted in the nonlinear multidisciplinary simulation software FEDEM, showing that the derived theory concur with the experimental results.

Tab. 1 Properties, damping ratio and eigenfrequency for the six different scenarios, based on analytical calculations (which are shown in the appendix).

Scenario	k [N/m]	K_p [N/m]	c [Ns/m]	K_d [Ns/m]	Damping ratio	Eigenfrequency
Undamped						
I)	100	0	0	0	$\zeta = 0$	$\omega_e = 1.59$ Hz
II)	0	44	0	0	$\zeta = 0$	$\omega_e = 1.06$ Hz
III)	100	44	0	0	$\zeta = 0$	$\omega_e = 1.91$ Hz
Damped						
IV)	100	44	7.2	0	$\zeta = 0.3$	$\omega_{e,d} = 1.82$ Hz
V)	100	44	0	4.8	$\zeta = 0.2$	$\omega_{e,d} = 1.87$ Hz
VI)	100	44	7.2	4.8	$\zeta = 0.5$	$\omega_{e,d} = 1.65$ Hz

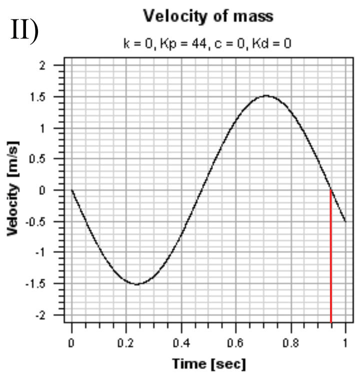
Undamped

I)



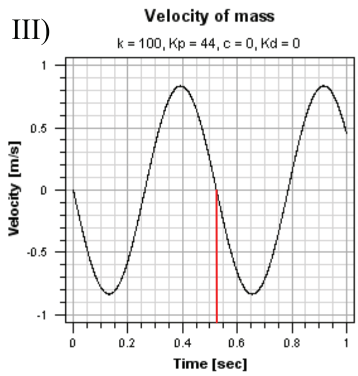
$t = 0.6285$ sec
$\Rightarrow \omega_e = 1.59$ Hz

II)



$t = 0.9475$ sec
$\Rightarrow \omega_e = 1.06$ Hz

III)

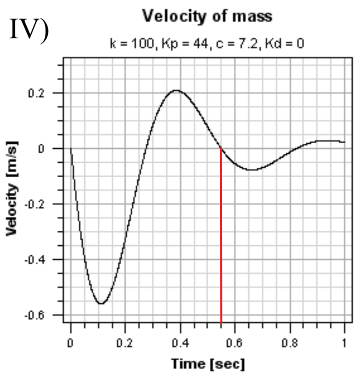


$t = 0.5240$ sec
$\Rightarrow \omega_e = 1.91$ Hz

Fig. 8 Results from FEDEM for the three undamped scenarios.

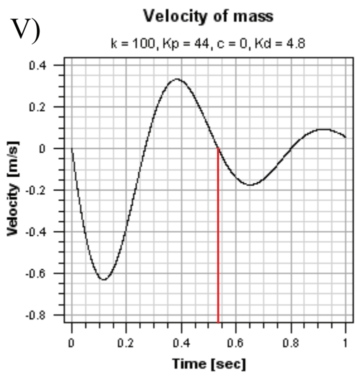
Damped

IV)



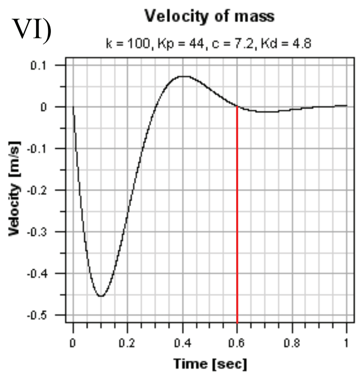
$t = 0.5490$ sec
$\Rightarrow \omega_{e,d} = 1.82$ Hz

V)



$t = 0.5345$ sec
$\Rightarrow \omega_{e,d} = 1.87$ Hz

VI)



$t = 0.6045$ sec
$\Rightarrow \omega_{e,d} = 1.65$ Hz

Fig. 9 Results from FEDEM for the three damped scenarios.

6 Acknowledgements

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8 Appendix

Below are shown the calculations for the eigenfrequencies in the six different experiment scenarios. The formulas used are: Eq. (26) for the undamped eigen-frequencies, Eq. (27) for the damped eigenfrequencies and Eq. (28) for the damping ratios.

Undamped

I) $\omega_e = \sqrt{\frac{(k+K_p)}{m}} = \sqrt{\frac{(100+0)}{1}} =$
$10 \text{ rad/sec} = 1.59 \text{ Hz}$

II) $\omega_e = \sqrt{\frac{(k+K_p)}{m}} = \sqrt{\frac{(0+44)}{1}} =$
$6.63 \text{ rad/sec} = 1.06 \text{ Hz}$

III) $\omega_e = \sqrt{\frac{(k+K_p)}{m}} = \sqrt{\frac{(100+44)}{1}} =$
$12 \text{ rad/sec} = 1.91 \text{ Hz}$

Damped

IV) $\zeta = \frac{(c+K_d)}{2\left(\sqrt{(k+K_p)m}\right)} = \frac{(7.2+0)}{2\left(\sqrt{(100+44)1}\right)} = 0.3$
$\Rightarrow \omega_{e,d} = \sqrt{\frac{(k+K_p)}{m}} \cdot \sqrt{1-\zeta^2} =$
$\sqrt{\frac{(100+44)}{1}} \cdot \sqrt{1-0.3^2} =$
$11.44 \text{ rad/sec} = 1.82 \text{ Hz}$

V) $\zeta = \frac{(c+K_d)}{2\left(\sqrt{(k+K_p)m}\right)} = \frac{(0+4.8)}{2\left(\sqrt{(100+44)1}\right)} = 0.2$
$\Rightarrow \omega_{e,d} = \sqrt{\frac{(k+K_p)}{m}} \cdot \sqrt{1-\zeta^2} =$
$\sqrt{\frac{(100+44)}{1}} \cdot \sqrt{1-0.2^2} =$
$11.76 \text{ rad/sec} = 1.87 \text{ Hz}$

VI) $\zeta = \frac{(c+K_d)}{2\left(\sqrt{(k+K_p)m}\right)} = \frac{(7.2+4.8)}{2\left(\sqrt{(100+44)1}\right)} = 0.5$
$\Rightarrow \omega_{e,d} = \sqrt{\frac{(k+K_p)}{m}} \cdot \sqrt{1-\zeta^2} =$
$\sqrt{\frac{(100+44)}{1}} \cdot \sqrt{1-0.5^2} =$
$10.39 \text{ rad/sec} = 1.65 \text{ Hz}$