Modeling and simulation of hybrid, multi-domain systems using switched bond graphs.

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Abstract:
When performing simulation of hybrid, multi-domain systems, the common analogies between the domains, like force-voltage-pressure and velocity-current-flow have to be extended to allow simulation of discrete events. When an event occurs, a law of conservation is used to determine variable values after the event. Then there is an analogy between,e.g., momentum conservation, charge conservation and volume conservation.
A bond graph is a tool for modeling and simulation of continuous multi-domain systems, that exploits the analogies between the domains. The extension to hybrid systems, switched bond graphs, also exploits the analogies that occur at discrete events, and can therefore be used for straightforward modeling and efficient simulation of hybrid, multi-domain systems.

1 Introduction

Hybrid system is a term used in many different areas to describe systems combining different properties. In this context, hybrid systems are systems that are both continuous and discrete to their nature [2, 8].

This paper deals with the problem of modeling and simulating hybrid systems [1, 5, 4]. The tool used for describing the hybrid systems is switched bond graphs [12, 4], which is a tool for making graphical physical models of multi-domain hybrid systems. The tool originates from (classical) bond graphs, which are used to model continuous systems.

In addition to the principle of energy conservation used when describing a system with a classical bond graph, a generalization of the principle of charge conservation is used to find the correct initialization of new modes and to detect discontinuities appearing when changing modes.

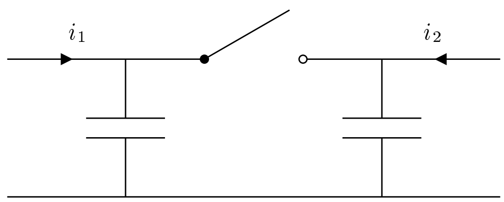

Figure 1: *A simple electrical example.*

Example 1 *Consider the two capacitors in Figure 1. When the switch is closed, their behavior is determined by the principle of electric charge conservation, stating that the amount of charge over the two capacitors is kept constant. This principle should be used to initialize the new mode. In this paper the charge conservation is generalized in a bond graph framework.*

When making physical models of continuous systems, the standard modeling procedure [12] is to repeatedly divide the system into subsystems, until each subsystem easily is described mathematically. The models of the subsystem are then composed to a model describing the complete system. This procedure should be possible to follow also when making models of hybrid systems. The aim is to find a modeling procedure where it is possible to describe the discrete phenomena locally, and still achieve a model that includes a proper description of non-local properties like the charge conservation in Example 1.

In this paper we will argue that switched bond graph supports such a modeling procedure.

2 Hybrid systems

The hybrid systems considered in this paper are continuous systems that are extended with discrete behavior. They arise from the natural process of model simplification. When making a model of a system, details that are considered to be unnecessary are left out in order to achieve a model that describes the interesting behavior of the system, without being to complex. Examples of mechanisms and phenomena that can be modeled with instant changes when their behavior is simplified are valves, diodes, mechanical collisions, *etc.* If, e.g., a valve is placed in a systems where other parts are modeled with continuous relations, the system is hybrid.

A mathematical description of such a hybrid systems is the *mode transition system* [12], see Figure 2. In a mode transition system there is a set of discrete states, sometimes called modes, each containing a continuous description. There are transitions that describe when a change of mode occurs, and to which mode the change is made. There is also a set of initialization rules, telling what values the state variables should have when starting simulation of a mode.

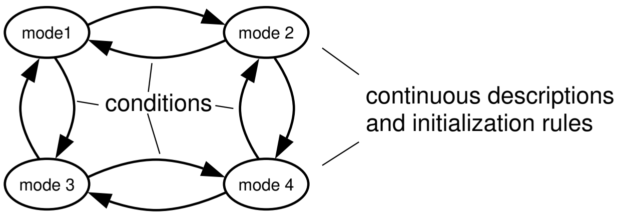

Figure 2: *A mode transition system.*

3 Analogies between different physical domains

When modeling multi-domain systems it is possible to get a better understanding of the systems by using analogies between the different domains. Since all systems are energy conserving, energy interactions are the most important analogies between the domains. Roughly speaking there are four different types of energy interactions in a system:

- energy can be entered to the system
- energy can be stored in the system
- energy can be transformed in the system
- energy can be dissipated from the system

The difference between the two last items can be subtle and depends on the modelers view of the system. Dissipated energy is energy transformed to a form that will not affect the interesting behavior of a system. For example, the heat dissipated from a resistor in an electrical circuit might not be of interest, hence the energy is dissipated. If the dissipated heat is of interest, it is instead a transformation of energy, from electrical energy to thermal.

To describe a system more in detail, we have to introduce variables that describe the energy interaction. A common way is to introduce two variables for each domain, where one describes the flow of the energy and the other describes the effort contained in the flow. In Table 1 some examples from different physical domains are shown.

Domain	Effort	Flow
mechanical	force	velocity
electrical	voltage	current
rot. mechanic	moment	angular velocity
hydraulic	pressure	flow

Table 1: *Effort and flow variables in different domains*

The variables together describe the energy flow of the system since their product quantity, effort $\times$ flow, is power.

When modeling a system that includes components that can be modeled as being discrete, the idealization of such components lead to a model where the energy conservation may not hold at the time instants of a discrete change.

Example 2 *Consider again the electrical example in Figure 1. When the switch closes, the charges will instantly be redistributed between the* *two capacitors. The total energy stored in the two capacitors before the switch closes is:*

$$E_1 = \frac{Q_a^2}{2C_a} + \frac{Q_b^2}{2C_b}$$

where Q_a and C_a is the charge and the capacitance of the first capacitor respectively, and Q_b and C_b the corresponding quantities of the second capacitor. The total energy after the switch is closed is:

$$E_2 = \frac{(Q_a + Q_b)^2}{2(C_a + C_b)}$$

It is straightforward to show that

$$E_1 \geq E_2$$

with equality if and only if

$$\frac{Q_a}{C_a} = \frac{Q_b}{C_b} \qquad \Leftrightarrow \qquad U_a = U_b$$

i.e., if the voltages across the two capacitors are the same before closing the switch.

To be able to handle instant changes, and account for the energy losses at the discrete changes of the system, we have to extend the continuous system analogies with a generalization of charge conservation, that includes, e.g., momentum conservation for mechanical systems. This conservation property is held by quantities that in [7, 6] are referred to as substance-like, since they are contained within a region of space. We will therefore call the generalization of charge conservation *substance*, see Table 2.

Domain	Substance	Appearance
electrical	charge	parallel conn. of capacitors
	magn. flux	series conn. of inductors
mechanical	momentum	conn. of masses
	displacem.	conn. of springs
hydraulical	volume	conn. of tanks

Table 2: *Examples of substances in different domains.*

It is also noticeable that the substance is found as the integral of either an effort variable or a flow variable, as shown by the following examples:

$$\text{charge} = \int \text{current } dt$$
$$\text{flux} = \int \text{voltage } dt$$
$$\text{displacement} = \int \text{velocity } dt$$
$$\text{volume} = \int \text{flow } dt$$

4 Switched bond graphs

The bond graph language [9, 10, 3] is a tool for physical modeling of systems which manifests the law of energy conservation. A bond graph is a graphical model which shows the power flow through the system. For electrical systems the bond graphs are similar to electrical circuit schemes, but bond graphs can also be used to model other types of systems like mechanical and hydraulic ones, and combinations of these.

The language contains a few elements, each describing how the energy in the system behaves. There are sources where the energy is entered, storage elements where energy can be stored, resistive elements where energy is dissipated, and transforming elements where the energy is transformed and a possible domain change is made.

To make the description complete, relations describing the behavior in detail have to be added to each of the elements. In a bond graph, a pair of variables is used to describe these mathematical relations; an effort, e, and a flow, f variable. The product of the two variables is the quantity power. In the mechanical domain, force is the effort variable and velocity the flow variable and in the electrical domain effort is voltage and flow is current. Hence the analogies from Section 3 are being used.

There are two types of junctions that describe how the elements are connected. They correspond to series and parallel connection in electrical circuits. They state that either the flows are equal and the efforts sum to zero (series connection) or that the efforts are equal and the flows sum to zero (parallel connection).

Half arrows called energy bonds are used to graphically describe the direction of the energy flow of the system. On a mathematical level the direction of the energy bonds determines the signs of the variables at the relations described by a junction.

Example 3 *There is a bond graph model of an electrical circuit in Figure 3. The* **C**- *and* **I**-*elements are storage elements, the* **Se**-*element is a source, the* **R**-*element is a dissipative element (resistor) and the* **1**-*junction is a series connection.*

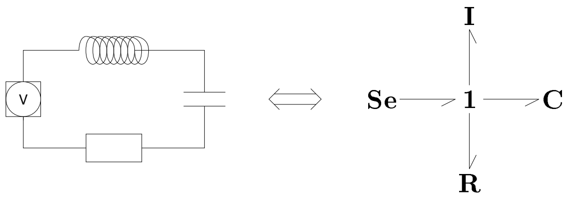

Figure 3: *A bond graph model of an electrical circuit.*

Example 4 *The circuit in Example 3 is a damped oscillation circuit. There is a mechanical equivalence shown in Figure 4. The bond graph model is the same, but in this case the* **C**-*element models the spring, the* **I**-*element models the mass, the* **Se**-*element models the force, and the* **R**-*element models the damper as illustrated in Figure 4. The* **1**-*junction states that the velocities of all elements are the same.*

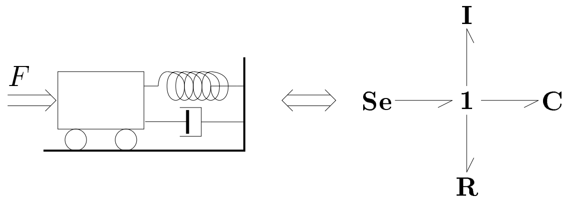

Figure 4: *A bond graph model of a mechanical system.*

An advantage with bond graphs is the possibility to algorithmically, and graphically, determine the causality [14, 9], i.e., in what order to calculate the variables to get as close to a state space description as possible. It is also algorithmic to derive equations from a bond graph [14, 9]. The equations can then be used for simulation or for formal analysis of the system.

Switched bond graphs [12, 4, 13] are an extension of classical bond graphs in the sense that classical bond graphs support modeling of continuous systems, while switched bond graphs also allow modeling of hybrid systems. When using switched bond graphs, the bond graph language is extended with a new element, a switch [13, 12]. The switch captures the discrete behavior with its two different states, denoted $\mathcal{E}$ and $\mathcal{F}$. Fixed in the $\mathcal{E}$-state the switch is replaced with a zero effort source, **Se**:0, and in the $\mathcal{F}$-state the switch is replaced with a zero flow source, **Sf**:0, see Figure 5. The causal assignment rules for the zero effort and flow sources are the same as for the ordinary sources.

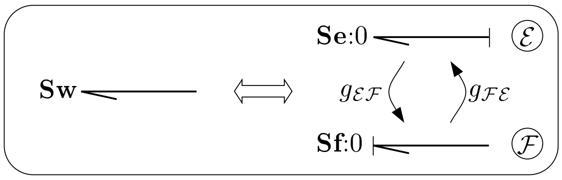

Figure 5: *The two states of a switch element.*

Note that a switched bond graph can be viewed as a collection of classical bond graphs, each describing a certain mode. To find such a classical bond graph, all switches have to be fixed in one of the two states, i.e., replaced with sources. There is one classical bond graph for each combination of switch states.

To complete the description of a switch, two transition conditions, $g_{\mathcal{E}\mathcal{F}}$ and $g_{\mathcal{F}\mathcal{E}}$, have to be added. $g_{\mathcal{E}\mathcal{F}}$ gives the condition for switching from state $\mathcal{E}$ to $\mathcal{F}$, and $g_{\mathcal{F}\mathcal{E}}$ the condition for switching the other way.

Example 5 *Connection and disconnection of two rigid bodies, as shown in Figure 6, can be modeled using a switch. The* $\mathcal{E}$-*state of the switch models the case when the two bodies do not have contact, and hence the force between them is zero. The* $\mathcal{F}$-*state of the switch models the case when the the bodies are connected and hence have a zero relative velocity.*

The condition for the bodies to loose contact is when the force between the becomes zero, hence

$$g_{\mathcal{F}\mathcal{E}} = [e \leq 0]$$

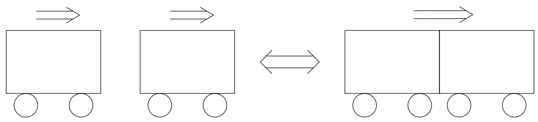

Figure 6: *Connection and disconnection of two wagons.*

and they meet when their relative position is zero:

$$g_{\mathcal{EF}} = [\int f dt \leq 0]$$

Since the bond graph language is build up by a few standard elements, there is a lot of structure in the models. In the next section, Section 5, it is argued that this structure is useful for the re-initializations after instant changes in a system during a simulation run. The structure in the graph is also very helpful when analyzing the model, e.g., when tracing errors in the model after an erroneous simulation run. It is the graphical causality algorithm that allows this analysis.

On the other hand, the structure in the bond graph limits the expressibility. Bond graphs does not have the same expressive power as e.g., object orientation.

Another important aspect of switched bond graphs is that they allow a structural change in the mathematical description when changing modes. The order in which the variables are calculated may change, and even the number of states can differ between the modes. This is due to the causal rules of the switch. For simulation purposes this means that a switched bond graph simulation tool allows simulation of a larger class of hybrid systems than tools that do not allow structural changes.

5 Substance in a switched bond graph

As noted, the analogies between different domains for continuous systems are explicit in a bond graph. In this section we will argue that the analogies at discrete changes also are explicit in the switched bond graph framework. First we will discuss discontinuities in a bond graph using specific domains, and then we will generalize the discussion to hold for any switched bond graph, independent of domain.

In a bond graph framework, a mass is modeled with an **I**-element. The mathematical relation corresponding to a linear **I**-element is [9, 3, 4]:

$$\dot{x} = e \tag{1}$$

$$f = \frac{1}{I} x \tag{2}$$

In a mechanical system the e variable is the force F, and the f variable is the velocity v. The parameter I is the mass m. There is also a state variable x introduced as the integral of the force according to Equation (1) and the discussion in Section 5. When rewriting Equations (1) and (2) in the domain specific variables, the following equations are obtained:

$$\dot{x} = F$$
$$v = \frac{1}{m} x$$

When looking at the second equation above, we see that the state variable in the bond graph is the momentum of the mass.

A capacitor is modeled with a bond graph **C**-element. The mathematical relation describing a **C**-element is the following:

$$\dot{x} = f$$
$$e = \frac{1}{C} x$$

In an electrical system the e variable is voltage u, the f variable current i, and the parameter C is the capacitance C. The corresponding equations using domain dependent notation are:

$$\dot{x} = i$$
$$u = \frac{1}{C} x$$

Here the state is the integral of the current, i.e., the charge.

We see that the entities that are preserved at instant changes in a system, e.g., electrical charge and momentum, appear as state variables in a bond graph framework. Hence a bond graph state variable describes substance.

The laws of charge conservation and momentum conservation can therefore be generalized in a bond graph frame work.

Remark 1 *The substance of a storage element in a bond graph is described by its state variable.*

In [11] the phenomena is referred to as *conservation of states*.

The substance can be used in exactly the same way as momentum or charge when instant changes take place in a system.

Postulate 1 *For instant changes in a bond graph the substance is kept constant.*

In a mechanical system the principle of momentum conservation states that the momentum

$$G = \sum m_k v_k = \sum \int F_k \tag{3}$$

is always preserved when there are instant changes in a mechanical system, e.g., when masses collide. The sum is over the colliding masses. The principle of electric charge conservation states that the charge

$$Q = \sum c_k u_k = \sum \int i_k \tag{4}$$

is always preserved after instant changes in an electrical system, e.g., when connecting capacitors in parallel. Here the sum is over the capacitors in the parallel connection.

If this is generalized in a bond graph framework, using the notion of substance, a conservation relation will have the following form:

$$S = \sum d_k x_k \tag{5}$$

where x_k is a state variable at some storage element in the bond graph. If the values of the coefficients d_k can be found from the bond graph, together with the information about what state variables that should be included in the sum, the substance relations can be found from the bond graph representation.

If causality is propagated in the mode specific bond graph corresponding to the mode entered at the discrete change, dependencies between storage elements can be found. These dependencies indicate what state variables have to be recalculated simultaneously to get proper simulation results. The dependencies can also be used to derive the relations needed to calculate the state variable values, i.e., to derive the coefficients in Equation 5. The procedure is described in detail in [4].

6 Example 1

Consider the electrical example in Figure 7. The bond graph model of the circuit is depicted in Figure 8. Assume that we want to simulate the model, starting with the switch open, and at a certain point of time close the switch.

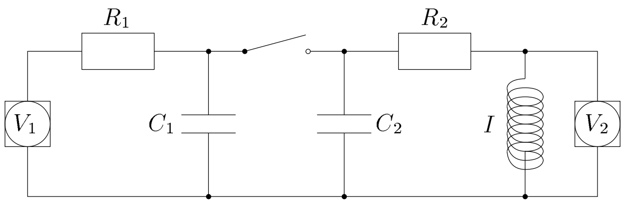

Figure 7: *An electrical example.*

To get a complete model, relations have to be added locally at each bond graph element. Assuming that all relations are linear, i.e., that we have linear electrical components, we only have to add the resistances, capacitances, etc. The switch conditions have to be added, e.g., that the switch closes at $t = 5$. Initial conditions for the continuous state variables also have to be added.

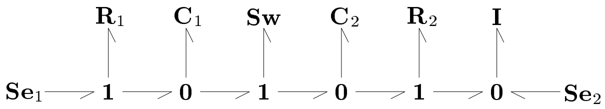

Figure 8: *A switched bond graph model of the electrical example.*

With this information the simulation is algorithmic from the bond graph [4]. First equations describing the system with the switch open are generated by a standard bond graph equation generation algorithm [14, 9] for the bond graph where the switch is replaced with a zero flow source, **Sf**:0.

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{R_1}V_1 - \frac{1}{R_1 C_1}x_1 \\
\dot{x}_2 &= \frac{1}{R_2}V_2 - \frac{1}{R_2 C_2}x_2 \\
\dot{x}_3 &= V_2
\end{aligned}$$

These equations are sent to a solver, together with the transition conditions. There is a zero-crossing detector in the numerical solver, detecting when a transition condition is met, in this case when $t = 5$. When this happens, the simulation stops and the new mode is entered.

The new mode has to be initialized. One initialization equation is achieved by the substance conservation, stating that the electrical charge over the two capacitors should be kept constant. In this case the transition take place at time $t = 5$. The value at time $t = 5$ before initialization is denoted by $x(5^-)$ and after by $x(5^+)$.

$$x_1(5^+) + x_2(5^+) = x_1(5^-) + x_2(5^-) \qquad (6)$$

To be able to initialize x_1 and x_2, one more equation is needed. This equation is found by deriving the equations for the second mode where the switch is closed, i.e., from the bond graph where the switch is replaced with a zero effort source:

$$\begin{aligned} \dot{x}_1 &= \frac{1}{R_1}V_1 + \frac{1}{R_2}V_2 - (\frac{1}{R_1C_1} + \frac{1}{R_2C_1})x_1 - \dot{x}_2 \\ x_2 &= \frac{C_2}{C_1}x_1 \\ \dot{x}_3 &= V_2 \end{aligned} \qquad (7)$$

Among these equations there is a static equation between x_1 and x_2 stating that the voltages over the two conductors should be the same. This equation has to hold initially.

Equations (6) and (7) are two equations in two unknowns that can be solved in order to initialize the second mode:

$$\begin{aligned} x_1(5^+) + x_2(5^+) &= x_1(5^-) + x_2(5^-) \\ x_2(5^+) - \frac{C_2}{C_1}x_1(5^+) &= 0 \end{aligned}$$

These equations can be generated and solved automatically from a switched bond graph [4].

When the second mode is properly initialized, the simulation continues, using the equations describing the second mode.

It can be noted that the equations describing mode two are not in state space form. The reason for this is that two storage elements are dependent in that mode. When closing the switch, the number of states is reduced. The reason for using three equations, when only two are needed for a state space description is to get a description where the substance corresponding to a certain storage element is easy to find.

We can also see that the information that is added locally at the different elements in the switched bond graph model, together with the description of how the elements are connected, is enough to derive the initial values of a new mode. No additional, global information has to be added to complete the model.

7 Example 2

In this section we will give a multi-domain example. Therefore, consider the DC-motor shown in Figure 9, and the corresponding switched bond graph model in Figure 10.

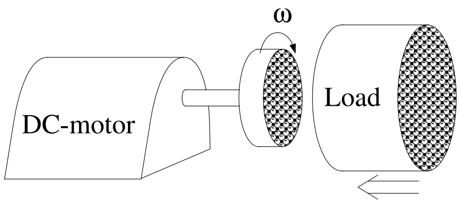

Figure 9: *A DC-motor.*

We will assume that the electrical part of the DC-motor contains inductance and resistance, modeled by $\mathbf{I}_1$ and $\mathbf{R}_1$ respectively, and a voltage source modeled by the **Se**-element. The moment of the motor is assumed to be proportional to the current, which is modeled by the **GY**-element. On the mechanical side we assume that there is friction and inertia, modeled by $\mathbf{I}_2$ and $\mathbf{R}_2$. We also assume that at time $t = 0.6$ a load is attached to the DC-motor. To simplify the model we assume that the load immediately becomes stuck to the DC-motor. Hence the attachment of the load can be modeled by a switch that includes the transition condition $g_{\mathcal{EF}} = [t = 0.6]$ and the load by an inertia as shown in Figure 10.

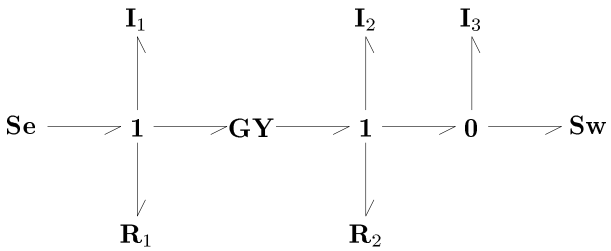

Figure 10: *A bond graph model of the DC-motor*

We also assume that that the system starts in rest, that the input voltage is 1 V and that all relations are linear with the following coefficients:

Coeff.	Quantity	Value
I_1	Inductance	$2\,H$
I_2	Inertia	$2\,kg\,m^2$
I_3	Inertia of load	$5\,kg\,m^2$
R_1	Resistance	$21\,\Omega$
R_2	Friction	$1\,N\,s/m$
K	Torque constant	$30\,Nm/A$

Given the information above, i.e., the graph, the relations, the parameter values, the transition conditions, the input voltage, and the initial state values, it is algorithmic to simulate the system

The first step in the simulation procedure is to derive equations for the case when the load is not connected. This is done by replacing the **Sw**-element in the bond graph in Figure 10 with a zero values effort source, **Se**: 0. Then equations are generated from the achieved bond graph:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}-\frac{R_1}{I_1} & -\frac{K}{I_2} & 0\\ \frac{K}{I_1} & -\frac{R_2}{I_2} & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix}1\\ 0\\ 0\end{pmatrix} u$$

$$g_{\mathcal{EF}} = [t = 0.6]$$

The equations are simulated until the transition condition is met. Then equations used to initialize the new mode are derived:

$$\begin{aligned} x_1(0.6^+) &= x_1(0.6^-)\\ x_2(0.6^+) + x_3(0.6^+) &= x_2(0.6^-) + x_3(0.6^-)\\ \frac{1}{I_2}x_2(0.6^+) - \frac{1}{I_3}x_3(0.6^+) &= 0 \end{aligned}$$

These equations are solved and the solution is the initial state value when simulating the system with the load attached.

To derive the equations when the load is attached, the **Sw**-element in the bond graph in Figure 10 is replaced by a zero values effort source, **Se**: 0.

$$\begin{aligned} \dot{x}_1 &= -\frac{R_1}{I_1}x_1 - \frac{K}{I_2}x_2\\ \dot{x}_2 &= \frac{K}{I_1}x_1 - \frac{R_2}{I_2}x_2 - \dot{x}_3\\ x_3 &= \frac{I_3}{I_2}x_2 \end{aligned}$$

Results from a simulation run is shown in Figures 11 and 12. In Figure 11, the angular velocity of the DC-motor is shown, and in Figure 12 the state variables of the system are plotted.

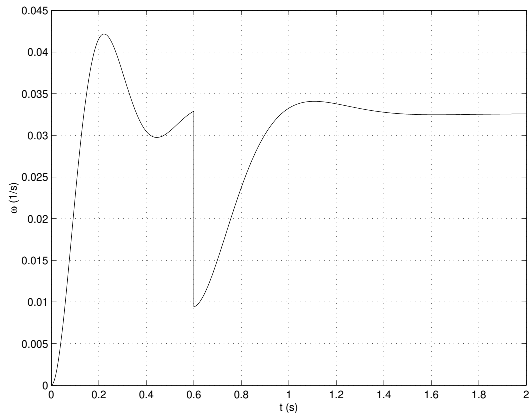

Figure 11: *The angular velocity of the DC-motor.*

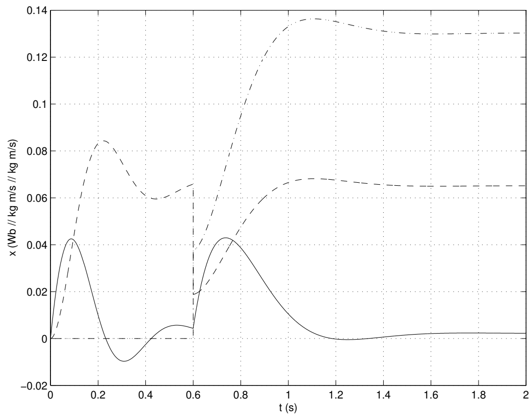

Figure 12: *The states of the system. Solid line: state x_1, the magnetic flux of the inductor. Dashed line: state x_2, the momentum of the mechanical part of the DC-motor. Dash-dotted line: state x_3, the momentum of the load.*

8 Conclusions

Switched bond graphs are natural extensions of classical bond graphs and they allow modeling of hybrid systems. They also give a generalization of the mechanical momentum that can be used for, e.g., simulation of hybrid systems. This extension, called substance, includes, e.g., charge in electrical circuits and volume in hydraulical systems.

In the same way as the principle of momentum conservation is used in mechanical systems to describe instant changes, substance conservation can be used in switched bond graphs to determine initial conditions after changing modes. Furthermore, the mode initialization procedure is algorithmic from a switched bond graph.

Hence switched bond graphs can be used to model and simulate hybrid systems in a straightforward manner.

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